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COMPARISON BETWEEN THE CONVERGENCE OF POWER AND GENERALIZED MITTAG-LEFFLER SERIES

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Abstract

In this paper we consider a family of the three-index generalizations of the classical Mittag-Leffler functions, introduced by Prabhakar. We consider series in such type of functions in the complex plane and study their convergence. More precisely, we determine where the series converges and where it does not, where the convergence is uniform, which the domain of convergence is, what the behaviour of the series is "near" the boundary of the domain of convergence, and on itself. Along with this, we state analogues of the Cauchy-Hadamard, Abel and Fatou theorems for the power series. Finally, we compare the obtained results with the classical ones for the widely used power series.

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1. Introduction

Let $E^{\gamma}_{\alpha,\beta}$ denote the Prabhakar generalization (see [13]) of the Mittag-Leffler (M-L) functions E_{α} and $E_{\alpha,\beta}$, defined in the whole complex plane $\mathbb{C}$ by the power series:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!}, \qquad \alpha,\ \beta,\ \gamma \in \mathbb{C}, \quad Re(\alpha) > 0, \tag{1.1}$$

where $(\gamma)_k$ is the Pochhammer symbol ([1], Section 2.1.1)

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma+1)\dots(\gamma+k-1).$$

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For $\gamma = 1$ this function coincides with M-L function $E_{\alpha,\beta}$, while for $\gamma = \beta = 1$ with E_α, i.e.:

$$E^1_{\alpha,1}(z) = E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \tag{1.2}$$

with $\alpha,\ \beta \in \mathbb{C}, \quad Re(\alpha) > 0$.

Consider now Prabhakar's generalization for indices $\beta = n$ with integer $n = 0, 1, 2, ...$, i.e.

$$E^\gamma_{\alpha,n}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+n)} \frac{z^k}{k!}, \qquad \alpha,\ \gamma \in \mathbb{C}, \quad Re(\alpha) > 0, \quad n \in \mathbb{N}_0. \tag{1.3}$$

Depending on γ and n, some coefficients in (1.3) may be equal to zero. This is possible only when $n = 0$ or γ is a non-positive integer. In the first case the coefficient with $k = 0$ is equal to zero, whereas in the second case (1.3) reduces to a finite sum, i.e. polynomial.

So, given a number γ, suppose that some of the coefficients in $E^\gamma_{\alpha,n}(z)$ are equal to zero, that is, there exist numbers $p, M \in \mathbb{N}_0$, such that the functions (1.3) can be written as follows:

$$E^\gamma_{\alpha,n}(z) = z^p \sum_{k=p}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+n)} \frac{z^{k-p}}{k!} \ \text{ or } \ E^\gamma_{\alpha,n}(z) = z^p \sum_{k=p}^{M} \frac{(\gamma)_k}{\Gamma(\alpha k+n)} \frac{z^{k-p}}{k!}. \tag{1.4}$$

More precisely, as it is seen above, if γ is different from zero, then $p = 1$ for $n = 0$, whereas $p = 0$ for each positive integer n. In the case $\gamma = 0$, the following remark can be made.

Remark 1.1. If $\gamma = 0$, then the functions in (1.3) take the simplest form

1. $E^0_{\alpha,n}(z) = 0$ for $n = 0$,
2. $E^0_{\alpha,n}(z) = \dfrac{1}{\Gamma(n)}$ for $n \in \mathbb{N}$.

Furthermore, an asymptotic formula for "large" values of the indices n is valid as follows, for a proof it could be seen in [10].

Theorem 1.1. *Let $z, \alpha, \gamma \in \mathbb{C}, n \in \mathbb{N}_0, Re(\alpha) > 0, \gamma \neq 0$. Then there exist entire functions $\theta^\gamma_{\alpha,n}$ such that the generalized Mittag-Leffler function* (1.3) *has the following asymptotic formula*

$$E^\gamma_{\alpha,n}(z) = \frac{(\gamma)_p}{\Gamma(\alpha p+n)} z^p \left(1 + \theta^\gamma_{\alpha,n}(z)\right), \tag{1.5}$$

where $\theta^{\gamma}_{\alpha,n}(z) \to 0$ *as* $n \to \infty$*, with a corresponding* p*, depending on the index* n*. Moreover, on the compact subsets of the complex plane* $\mathbb{C}$*, the convergence is uniform and*

$$\theta^{\gamma}_{\alpha,n}(z) = O\left(\frac{1}{n^{Re(\alpha)}}\right) \quad (n \ \in \mathbb{N}). \tag{1.6}$$

Remark 1.2. According to the asymptotic formula (1.5), it follows that there exists a positive integer N_0 such that the functions (1.3) have no zeros for $n > N_0$, possibly except for the origin.

Remark 1.3. Each of the functions in (1.3) ($n \in \mathbb{N}$), being an entire function, not identically zero, has no more than a finite number of zeros in the closed and bounded set $|z| \le R$. Moreover, because of Remark **1.2**, no more than finite number of these functions have some zeros, possibly except for the origin.

2. Series in generalized M-L functions

In this section we recall briefly some results on the convergence in the complex plane of series in generalized M-L functions, like these in (1.3). These are results quite analogous to the ones for the classical power series. The same type convergence theorems have been earlier obtained for series in some other special functions, for example, for series in Laguerre and Hermite polynomials, by Rusev ([14]), and resp. by the author – for series in Bessel functions, their Wright's 2-, 3-, and 4-indices generalizations, and also more general multi-index (in a sense of [3], [2]) M-L functions (see e.g. [5]–[9]).

Setting

$$\widetilde{E}^{0}_{\alpha,0}(z) = 0, \quad \widetilde{E}^{0}_{\alpha,n}(z) = \Gamma(n)\ z^n\ E^{0}_{\alpha,n}(z), \quad n \in \mathbb{N},$$

$$\widetilde{E}^{\gamma}_{\alpha,n}(z) = \frac{\Gamma(\alpha p + n)}{(\gamma)_p}\ z^{n-p}\ E^{\gamma}_{\alpha,n}(z), \quad n \in \mathbb{N}_0 \quad (\gamma \ne 0), \tag{2.1}$$

(with the corresponding values of p), we consider the series in these functions, respectively of the form:

$$\sum_{n=0}^{\infty} a_n \widetilde{E}^{\gamma}_{\alpha,n}(z), \tag{2.2}$$

with complex coefficients a_n $(n = 0, 1, 2, ...)$.

Finding their disks of convergence, we study the series behaviour inside the found disks and "near" their boundaries, as well as on the boundaries, giving theorems of Cauchy-Hadamard, Abel, as well as Fatou type.

3. Cauchy-Hadamard and Abel type theorems

In the beginning, we state a theorem of Cauchy-Hadamard type and a corollary for the series (2.2), considered above.

In what follows we use the notation $D(0;R)$ and $C(0;R)$ respectively for the open disk centered at the origin with a radius R and its boundary, i.e.

$$D(0;R) = \{z : |z| < R,\ z \in \mathbb{C}\}, \quad C(0;R) = \partial D(0;R) = \{z : |z| = R,\ z \in \mathbb{C}\}.$$

Theorem 3.1 (of Cauchy-Hadamard type)**.** *The domain of convergence of the series* (2.2) *with complex coefficients* a_n *is the disk* $D(0;R)$ *with a radius of convergence* R*, where*

$$R = \left(\limsup_{n\to\infty} (\ |a_n|\ )^{1/n} \right)^{-1}. \tag{3.1}$$

More precisely, the series (2.2) *is absolutely convergent on the disk* $D(0;R)$ *and divergent on the domain* $|z| > R$*. The cases* $R = 0$ *and* $R = \infty$ *fall in the general case.*

Thus, the considered series (2.2) converges in a disk, like in the theory of the widely used power series. Analogously, inside the disk, the convergence of the discussed series is uniform, i.e., the following corollary, similar to the classical Abel lemma, holds.

Corollary 3.1. *Let the series* (2.2) *converges at the point* $z_0 \neq 0$*. Then it is absolutely convergent on the disk* $D(0;|z_0|)$*. Inside the disk* $D(0;R)$*, i.e. on each closed disk* $|z| \leq r < R$ *(*R *defined by* (3.1)*), the convergence is uniform.*

The very disk of convergence is not obligatory a domain of uniform convergence and on its boundary the series may even be divergent.

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angular domain with size $2\varphi < \pi$ and with a vertex at the point $z = z_0$, which is symmetric with respect to the straight line defined by the points 0 and z_0 and d_φ be the part of the angular domain g_φ, closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle. The next theorem refers to the uniform convergence of the series (2.2) on the set d_φ and its convergence at the point z_0, provided $z \in D(0;R) \cap g_\varphi$.

Theorem 3.2 (of Abel type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, R be the real number defined by* (3.1) *and* $0 < R < \infty$. *If $\widetilde{f}(z;\,\alpha,\,\gamma)$ is the sum of the series* (2.2) *on the domain $D(0;R)$, and this series converges at the point z_0 of the boundary $C(0;R)$, then:*

(i) *The following relation holds*

$$\lim_{z\to z_0} \widetilde{f}(z;\,\alpha,\,\gamma) = \sum_{n=0}^{\infty} a_n \widetilde{E}^{\gamma}_{\alpha,\,n}(z_0), \tag{3.2}$$

provided $z \in D(0;R) \cap g_\varphi$.

(ii) *The series* (2.2) *is uniformly convergent on the domain d_φ.*

The details of the proofs concerning the series (2.2) (including the equality (3.2)) could be seen in [10], except for the uniformity and Corollary. The ideas of the last ones go analogously to the [11].

4. Fatou type theorem

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers with $\limsup\limits_{n\to\infty} (|a_n|)^{1/n} = R^{-1}$, $0 < R < \infty$ and $f(z)$ be the sum of the power series $\sum\limits_{n=0}^{\infty} a_n z^n$ on the open disk $D(0;R)$, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in D(0;R). \tag{4.1}$$

Definition 4.1. A point $z_0 \in \partial D(0;R)$ is called regular for the function f if there exist a neighbourhood $U(z_0;\rho)$ and a function $f^*_{z_0} \in \mathcal{H}(U(z_0;\rho))$ (the space of complex-valued functions, holomorphic in the set $U(z_0;\rho)$), such that $f^*_{z_0}(z) = f(z)$ for $z \in U(z_0;\rho) \cap D(0;R)$.

By this definition it follows that the set of regular points of the power series is an open subset of the circle $C(0;R) = \partial D(0;R)$ with respect to the relative topology on $\partial D(0;R)$, i.e. the topology induced by that of $\mathbb{C}$.

In general, there is no relation between the convergence (divergence) of a power series at points on the boundary of its disk of convergence and the regularity (singularity) of its sum of such points. For example, the power series $\sum\limits_{n=0}^{\infty} z^n$ is divergent at each point of the unit circle $C(0;1)$ regardless of the fact that all the points of this circle, except for $z = 1$, are regular for its sum. The series $\sum\limits_{n=1}^{\infty} n^{-2} z^n$ is (absolutely) convergent at each point of the circle $C(0;1)$, but nevertheless one of them, namely $z = 1$, is a singular (i.e. not regular) for its sum. However, under additional conditions on the sequence $\{a_n\}_{n=0}^{\infty}$, such a relation does exists (see for details Fatou theorem in [4], Vol.1, Ch. 3, §7, 7.3, p. 357), namely, if the coefficients of the power series with the unit disk of convergence tend to the zero, i.e. $\lim\limits_{n\to\infty} a_n = 0$, then the power series converges, even uniformly, on each arc of the unit circle, all points of which (including the ends of the arc) are regular for the sum of the series.

Propositions referring to the properties discussed above have been established also for series in the Laguerre and Hermite polynomials, as well as in Mittag-Leffler systems (see e.g. [14], resp. [12]). Here we give such type of theorem for the Prabhakar systems as follows.

Theorem 4.1 (of Fatou type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the conditions*

$$\lim_{n\to\infty} a_n = 0, \qquad \limsup_{n\to\infty} (\,|a_n|\,)^{1/n} = 1, \tag{4.2}$$

and $F(z)$ be the sum of the series (2.2) *on the unit disk $D(0;1)$, i.e.*

$$F(z) = \sum_{n=0}^{\infty} a_n \widetilde{E}^{\gamma}_{\alpha,n}(z), \quad z \in D(0;1).$$

Let σ be an arbitrary arc of the unit circle $C(0;1)$ with all its points (including the ends) regular to the function F. Then the series (2.2) *converges, even uniformly, on the arc σ.*

The **Proof** follows the lines of this one for the Mittag-Leffler functions, using the asymptotic formula (1.5) (for details, see [12]). ■

5. Special cases

In particular, as it has been discussed in the Introduction, for $\gamma = 1$ the Prabhakar function $E^{\gamma}_{\alpha,\beta}$, defined by (1.1), coincides with M-L function $E_{\alpha,\beta}$, i.e. $E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$ (see (1.2)). So in this case the series (2.2) takes the form

$$\sum_{n=0}^{\infty} a_n \widetilde{E}^{1}_{\alpha,n}(z) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,n}(z), \tag{5.1}$$

with complex coefficients a_n $(n = 0, 1, 2, ...)$.

Such a kind of series were studied in details e.g. in [11] and [12], but all the obtained results concerning them follow as particular cases from the preceding sections, as well.

6. Conclusion

We emphasize that the results obtained for the series (2.2) are the same as these for the power series (4.1). As it is well seen, they have one and the same radius of convergence R, and are both absolutely and uniformly convergent on each closed disk $|z| \leq r$ ($r < R$). More precisely, if each one of them converges at the point z_0 of the boundary of $D(0; R)$, then the theorems of Abel type hold for both series in one and the same angular domain. Finally, if $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers satisfying the conditions (4.2), and all the points (including the ends) of the arc σ of the unit circle $C(0; 1)$ are regular to the sums of both considered series, then the series (2.2) and (4.1) converge even uniformly, on the arc σ.

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